View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 327 (2007) 869–877

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Integrability of rotationally symmetric n-harmonic maps

Chao-Nien Chen [a], L.F. Cheung [b], Y.S. Choi [c], C.K. Law [d,e,∗]

[a] Department of Mathematics, National Changhua University of Education, Changhua 500, Taiwan, ROC
[b] Department of Mathematics, The Chinese University of Hong Kong Shatin, Hong Kong
[c] Department of Mathematics, University of Connecticut, Storrs, CT 06269-3009, USA
[d] Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 804, Taiwan, ROC
[e] National Center for Theoretical Sciences, Taiwan, ROC

Received 14 October 2005

Available online 30 May 2006

Submitted by V. Radulescu

Abstract

A rotationally symmetric n-harmonic map is a rotationally symmetric p-harmonic map between two n-dimensional model spaces such that $p = n$. We show that rotationally symmetric n-harmonic maps can be integrated and are n-harmonic diffeomorphism, and apply such results to investigate the asymptotic behaviors of these maps. We also derive this integrability using Lie theory.
© 2006 Elsevier Inc. All rights reserved.

Keywords: p-Harmonic maps; n-Harmonic maps; Rotational symmetry; Model space; Dichotomy phenomenon

1. Introduction

The p-harmonic maps ($p \geqslant 2$) are the solutions of the Euler–Lagrange equations associated with p-energy $E(\phi)$ defined by

$$E(\phi) = \frac{1}{p} \int_M |d_x \phi|^p,$$

∗ Corresponding author.
E-mail addresses: chenc@math.ncue.edu.tw (C.-N. Chen), lfcheung@math.cuhk.edu.hk (L.F. Cheung), choi@math.uconn.edu (Y.S. Choi), law@math.nsysu.edu.tw (C.K. Law).

0022-247X/$ – see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2006.04.073

where ϕ is a C^1-map from a Riemannian manifold M to another Riemannian manifold N. A Riemannian manifold M is called a *model space* (see, e.g., [8]) if

$$M = M^n(f) \equiv \left(S^{n-1} \times [0, +\infty), f^2(r)\,d\varphi^2 + dr^2\right),$$

where $(S^{n-1}, d\varphi^2)$ is the $(n-1)$-dimensional Euclidean sphere. When a p-harmonic map ϕ between model spaces $M^n(f)$ and $N^n(g)$ satisfies the rotational symmetry condition

$$\phi(r, \varphi) = \left(y(r), \varphi\right) \quad \text{for all } \varphi \in S^{n-1},$$

then y satisfies the following ordinary differential equation [6]:

$$\left(\theta(r)^{\frac{p}{2}-1} y'\right)'(r) + (n-1)\theta(r)^{\frac{p}{2}-1}\frac{f'(r)}{f(r)}y'(r) - (n-1)\theta(r)^{\frac{p}{2}-1}\frac{g(y(r))g'(y(r))}{f(r)^2} = 0, \tag{1.1}$$

such that $y(0) = 0$. Here

$$\theta(r) = y'(r)^2 + (n-1)\frac{g(y(r))^2}{f(r)^2}, \tag{1.2}$$

and the *warping functions* f, g satisfy the conditions

$$f, g \in C^1[0, \infty), \quad f(0) = g(0) = 0, \qquad f'(0) = g'(0) = 1, \quad f', g' > 0. \tag{1.3}$$

Throughout the paper, we shall assume that f and g satisfy (1.3). When the independent variable is not specified, $'$ denotes the derivative with respect to r, while $g'(y)$ denotes the derivative of the function g with respect to y. We shall also assume n, dimension of the domain manifold, to be greater than 1, for the case $n = 1$ is trivial.

Let $0 < R_0 \leqslant \infty$ and $y \in C^2(0, R_0) \cap C[0, R_0)$ be a positive function (i.e. $y > 0$ in $(0, R_0)$) satisfying $\lim_{r\to 0+} y(r) = 0$. In the following, by an abuse of language, we call y a *rotationally symmetric p-harmonic map* between model spaces $M^n(f)$ and $N^n(g)$. Each rotationally symmetric p-harmonic map stands for a p-harmonic map, with possibly infinite p-energy [6]. In this paper, we are interested in the case when $p = n$, which we call *rotationally symmetric n-harmonic maps*.

A p-harmonic map when $p = 2$ is usually called a harmonic map. In this case, (1.1) is reduced to

$$y''(r) + (n-1)\frac{f'(r)}{f(r)}y'(r) - (n-1)\frac{g(y(r))g'(y(r))}{f(r)^2} = 0. \tag{1.4}$$

Equation (1.4) was first systematically studied by Ratto and Rigoli [12]. See also [13,14]. It is easy to see [3] that when $n = 2$, (1.4) becomes

$$f(fy')' = g(y)g'(y)$$

which can be integrated to obtain

$$f(r)y'(r) = g\left(y(r)\right). \tag{1.5}$$

Equation (1.5) has a nice geometric interpretation, which is the conformality of the map. Conformality is known for harmonic maps even without rotational symmetry, provided that the harmonic maps are maps between 2-dimensional manifolds of genus zero.

In this note, it will be shown that all rotationally symmetric n-harmonic maps satisfy (1.5), and hence are conformal maps, except perhaps at the origin. In other words, the associated

n-harmonic maps can be parametrized by the same family of functions which are independent of n. This is a bit surprising but in fact not so in view of the conformal invariance of the associated p-energy functional. The interested reader may refer to [10, pp. 3–4] and a paper by D'Onofrio and Iwaniec [11] for an exposition of this well-known fact.

We shall show that (1.1) can be integrated to obtain (1.5) when $p = n$. Applying (1.5), we find that the inverse of a rotationally symmetric n-harmonic map is also a rotationally symmetric n-harmonic map. Thus each rotationally symmetric n-harmonic map stands for a harmonic diffeomorphism! For some special model spaces, we can solve (1.5) for closed form solutions.

Theorem 1.1. *Let $y \in C^1(0, R_0)$. If y satisfies* (1.5) *with $y(r_0) > 0$ for some $r_0 \in (0, R_0)$, then $y \in C^2(0, R_0) \cap C[0, R_0)$ with $y > 0$ in $(0, R_0)$ and $y(0) = \lim_{r\to 0^+} y(r) = 0$. Moreover it is a rotationally symmetric n-harmonic map.*

Proof. Let y be a solution of (1.5). It is clear that $y(r) \neq 0$ for all $r \in (0, R_0)$, otherwise the uniqueness of solution to ordinary differential equation requires y be identically zero on $(0, R_0)$, contradicting $y(r_0) > 0$ for some r_0. Hence y is a positive function. Since y is strictly monotone increasing, $\lim_{r\to 0^+} y(r)$ exists. Therefore we can assume $y \in C[0, R_0)$.

Regularity bootstrap on (1.5) gives $y \in C^2(0, R_0)$. From its definition (1.2), $\theta = ng(y)^2/f^2$. With $p = n$, it can then be checked that y satisfies (1.1) by simple calculations. Hence y is a rotationally symmetric n-harmonic map.

Next, let $y_0 = y(r_0)$. We can integrate (1.5) to obtain

$$\int_{y(r)}^{y_0} \frac{d\xi}{g(\xi)} = \int_{r}^{r_0} \frac{d\eta}{f(\eta)}.$$

From condition (1.3), the right-hand side of the above equation goes to infinity like $-\log r$ as $r \to 0$. Thus it is necessary for $\lim_{r\to 0^+} y(r) = 0$ in order for its left side to balance such a blow-up. The proof is complete. $\square$

Not only solutions of (1.5) are sufficient conditions for them to be solutions of (1.1), they are also necessary conditions as indicated by the following theorem.

Theorem 1.2. *Let $y \in C^2(0, R_0) \cap C[0, R_0)$ be a positive function. Then y is a rotationally symmetric n-harmonic map from $M^n(f)$ to $N^n(g)$ with $y(0) = 0$ if and only if it satisfies* (1.5). *Furthermore the inverse function of y is also a rotationally symmetric n-harmonic map from $N^n(g)$ to $M^n(f)$.*

Remark. Since it is well known that $\mathbb{R}^n$ and $\mathbb{H}^n$ are model spaces with warping functions r and $\sinh r$, respectively, we can solve for the corresponding rotationally symmetric n-harmonic maps. Thus the only family of rotationally symmetric n-harmonic maps is $y = Cr$ from $\mathbb{R}^n$ to $\mathbb{R}^n$; $y = C \tanh r/2$ from $\mathbb{H}^n$ to $\mathbb{R}^n$; $y = 2\tanh^{-1}(Cr)$ from $\mathbb{R}^n$ to $\mathbb{H}^n$; and $y = 2\tanh^{-1}(C \tanh r/2)$ from $\mathbb{H}^n$ to $\mathbb{H}^n$, respectively. The details are given in [3]. When the model space is the n-sphere S^n instead of $\mathbb{H}^n$, the function tan will appear in the map instead of tanh.

In [1], Chang, Ding and Ye proved an interesting result concerning the blow-up phenomenon for rotationally symmetric harmonic flow from a unit ball of $\mathbb{R}^2$ to S^2. Later Grotowski [9] extended the result to harmonic flow from a unit ball of $\mathbb{R}^3$ to S^2. We recently extended the blow-up result to the heat flow for 3-harmonic maps from a unit ball of $\mathbb{R}^3$ to S^3 [2]. There the family of rotationally symmetric n-harmonic maps obtained above played a crucial role in the construction of suitable subsolutions for the heat flow.

Another application of Theorem 1.2 is to understand how the warping functions of model spaces relates to the growth of rotationally symmetric n-harmonic maps. For $p \geqslant 2$, which may differ from n, a rotationally symmetric p-harmonic map y satisfies $y'(r) > 0$ for all $r > 0$ [3,6]. Thus y has one of the three kinds of asymptotic behaviors: bounded on $\mathbb{R}^+$, or blows up at a finite point, or goes to infinity as $r \to \infty$. If $p = n$, we can exploit the exact integrability in this case to classify this asymptotic behavior. It turns out these are determined by whether $1/f$ and $1/g$ lie in $L^1(1,\infty)$ or not. More precisely, we have the following theorem.

Theorem 1.3. *Let $y \in C^2(0, R_0) \cap C[0, R_0)$ be a rotationally symmetric n-harmonic map from $M^n(f)$ and $N^n(g)$ with $y(0) = 0$. Here $0 < R_0 \leqslant \infty$ is the maximal interval of existence for the solution y.*

(a) *If $1/f \in L^1(1,\infty)$ and $1/g \notin L^1(1,\infty)$, then $R_0 = \infty$, and y is bounded on $\mathbb{R}^+$.*
(b) *If $1/f \notin L^1(1,\infty)$ and $1/g \in L^1(1,\infty)$, then R_0 is finite, and y is unbounded on* $(0, R_0)$.
(c) *If both $1/f$ and $1/g$ do not lie in $L^1(1,\infty)$, then $R_0 = \infty$ and y is unbounded on $\mathbb{R}^+$.*
(d) *If both $1/f, 1/g \in L^1(1,\infty)$, then there exists a unique rotationally symmetric n-harmonic map y_β which behaves like in case* (c). *Rotationally symmetric n-harmonic maps lying above y_β behaves like in* (b), *while those lying under y_β behaves like in* (a). *All such maps do not intersect one another in the y versus r plane except at* $(r, y) = (0, 0)$.

Remarks.

(1) Let $f, g \in C^3[0,\eta)$. It follows naturally [4, p. 1253] from rotational symmetry that $f''(0) = g''(0) = 0$. Hence by [5, Theorem 2.1] and [4, Theorem 2.11] that for any rotationally symmetric n-harmonic map y, $y'(0)$ exists and $y \in C^1[0, R_0)$. In such a case, results in part (d) imply that there exists an $\alpha_0 > 0$ such that if $y'(0) > \alpha_0$, then y behaves as in case (b) in the above theorem. While if $y'(0) < \alpha_0$, it behaves as in case (a).
(2) We note that part (a) of the above theorem has been shown to hold for rotationally symmetric harmonic maps [4, Theorem 5.10].

The integrability of (1.1) demonstrates some kind of symmetry within the equation. We find that one can use Lie's theory to derive such an integrability. This derivation, though formal, seems to be interesting too.

We shall prove Theorem 1.2 in Section 2. Differential identities derived from (1.1) are crucial. Theorem 1.3 will be proved in Section 3. In Section 4, (1.5) is derived for the special case $f(r) = r$ using Lie's theory. This symmetry should be due to conformal invariance.

2. Rotationally symmetric n-harmonic maps

We recall that a rotationally symmetric n-harmonic map between the model spaces $M^n(f)$ and $N^n(g)$ involves finding a positive classical solution y to Eq. (1.1) with $p = n$ and $y(0) \equiv \lim_{r\to 0^+} y(r) = 0$ in the interval $[0, R_0)$. Here $0 < R_0 \leqslant \infty$ is the maximal interval of existence for such solution. Though y may not be in $C^1[0, R_0)$, its derivative cannot blow up too quickly near $r = 0$ in the following sense.

Proposition 2.1. *Let $y \in C^2(0, R_0) \cap C[0, R_0)$ be a positive solution to* (1.1) *with $p = n$. Then*

$$\lim_{r\to 0^+} f(r)y'(r) = 0.$$

Proof. First, we derive from (1.1) that

$$\left(f^{n-1}\theta^{\frac{n}{2}-1}y'\right)' = (n-1)f^{n-3}\theta^{\frac{n}{2}-1}g(y)g'(y) \geqslant 0.$$

This allows us to conclude that $f^{n-1}\theta^{n/2-1}y'$ is monotonically increasing. It therefore makes sense to define a finite number $\gamma \geqslant 0$ (as it is known that $y' > 0$ [6, Lemma 2.1]) satisfying

$$\gamma \equiv \lim_{r\to 0^+} f(r)^{n-1}\theta(r)^{\frac{n}{2}-1}y'(r). \tag{2.1}$$

When $n = 2$, (2.1) is reduced to $\lim_{r\to 0^+} f(r)y'(r) = \gamma$. If $\gamma > 0$, let $y'(r) = \gamma/r + o(1/r)$ for $r \leqslant r_0$, where $r_0 > 0$ is sufficiently small. Upon integration, $y = \gamma \log r + o(\log r)$ for $r \leqslant r_0$. This contradicts the hypothesis that y is bounded in $[0, r_0]$. Thus $\gamma = 0$.

When $n \geqslant 3$, we substitute (1.2) into (2.1) to obtain

$$\gamma = \lim_{r\to 0+} f(r)y'(r)\left(f(r)^2y'(r)^2 + (n-1)g\big(y(r)\big)^2\right)^{\frac{n}{2}-1}.$$

Hence

$$\limsup_{r\to 0+}\big(f(r)y'(r)\big)^{n-1} \leqslant \gamma \leqslant \liminf_{r\to 0+}\left(f(r)^2y'(r)^2 + (n-1)g\big(y(r)\big)^2\right)^{\frac{n-1}{2}}.$$

As $\lim_{r\to 0+} g(y(r)) = 0$, we have $\gamma^{\frac{1}{n-1}} = \lim_{r\to 0+} f(r)y'(r)$. With a similar argument as above, $\gamma = 0$. $\square$

Proof of Theorem 1.2. Let y be a positive solution of (1.5). By Theorem 1.1, $y(0) = 0$ and it is a rotationally symmetric n-harmonic map.

Conversely, let y be a rotationally symmetric n-harmonic map with $y(0) = 0$. Hence,

$$y'' + (n-1)\frac{f'}{f}y' + \left(\frac{p}{2} - 1\right)\frac{\theta'}{\theta}y' - (n-1)\frac{g(y)g'(y)}{f^2} = 0, \tag{2.2}$$

where we have suppressed the dependency of the various functions on r for notation simplicity. To eliminate θ from the above equation, we first differentiate (1.2) with respect to r,

$$\theta' = 2\left[y'y'' + (n-1)\frac{f^2g(y)g'(y)y' - ff'g(y)^2}{f^4}\right]. \tag{2.3}$$

Upon eliminating y'' from the above equation and (2.2), we obtain

$$\left[(p-1)y'^2 + (n-1)\frac{g(y)^2}{f^2}\right]\theta' = 2(n-1)\theta\left[\frac{2g(y)g'(y)y'}{f^2} - \frac{f'}{f}\left(y'^2 + \frac{g(y)^2}{f^2}\right)\right].$$

When $p = n$, it becomes

$$\theta' = 2\theta\left\{\frac{2g(y)g'(y)y'}{f^2y'^2 + g(y)^2} - \frac{f'}{f}\right\}. \tag{2.4}$$

Substituting (2.4) into (2.2) yields

$$y'' + \frac{f'}{f}y' = (n-1)\frac{g(y)g'(y)}{f^2} - 2(n-2)\frac{g(y)g'(y)y'^2}{f^2y'^2 + g(y)^2}.$$

We now multiply f^2y' on both sides and get

$$fy'(fy')' = (n-1)g(y)g'(y)y' - 2(n-2)\frac{f^2g(y)g'(y)y'^3}{f^2y'^2 + g(y)^2}.$$

This is equivalent to

$$\big(f^2y'^2 - g(y)^2\big)' = (n-2)\frac{g(y)^2 - f^2y'^2}{g(y)^2 + f^2y'^2}\big(g(y)^2\big)'. \tag{2.5}$$

Define $u = f(r)^2y'(r)^2 - g(y)^2$ and $v = g(y)^2$. With $y > 0$, we have $v > 0$ in the interior of the domain. Therefore $u + 2v > 0$ and (2.5) is equivalent to

$$(u + 2v)u' + (n-2)uv' = 0.$$

Multiplying the above equation by $u(u + 2v)^{n-3}$, it can be checked that its left-hand side can be readily integrated to obtain $u^2(u + nv)^{n-2} = C$ for some constant $C \geqslant 0$. Hence

$$\big(f^2y'^2 - g(y)^2\big)^2\big(f^2y'^2 + (n-1)g(y)^2\big)^{n-2} = C.$$

We can now use Proposition 2.1. Together with $\lim_{r\to 0+} g(y(r)) = 0$, we conclude that $C = 0$ by taking limit as $r \to 0^+$. Therefore, $f(r)y'(r) = g(y)$ has to hold. This proves the first part.

For the second part, we consider r as a function of y. Then $r'(y) = 1/y'(r)$ and so by (1.5), $r'(y)g(y) = f(r)$. □

3. Proof of Theorem 1.3

Define $r_0 = R_0/2$, and $y_0 = y(r_0)$. Since y satisfies (1.5) by Theorem 1.2,

$$\int_{y_0}^{y(r)} \frac{d\xi}{g(\xi)} = \int_{r_0}^{r} \frac{d\eta}{f(\eta)}. \tag{3.1}$$

Let $G(y(r))$ and $F(r)$ be the above integrals on the left-hand side and the right-hand side, respectively. Hence the above equation is equivalent to

$$G\big(y(r)\big) = F(r). \tag{3.2}$$

Equation (3.2) shows how the growth of y relates to the warping functions. The solution can be continued to $r = R_0$ when either R_0 or $y(R_0)$ blows up.

Without loss of generality, we can assume integrability of $1/f$ and $1/g$ on the interval $[1,\infty)$ is equivalent to that on the intervals $[r_0,\infty)$ and $[y_0,\infty)$, respectively.

(a) With the indicated integrability of $1/f$ and $1/g$ in this case, we have $M \equiv F(\infty)$ being finite and $G(\infty) = \infty$. From (3.2), $G(y(r)) = F(r) \leqslant M$ for all $r \in [0, R_0)$. Thus y is bounded on $[0, R_0)$. Since either R_0 or $y(R_0)$ blows up, we can conclude that $R_0 = \infty$.

(b) In this case, $F(\infty) = \infty$ and $M \equiv G(\infty)$ being finite. Thus by (3.2) $F(r) = G(y(r)) \leqslant M$ for all $r \in [0, R_0)$. Hence R_0 has to be finite. This leads us to conclude that $y(R_0) = \infty$.

(c) In this case, $F(\infty) = G(\infty) = \infty$. We know either R_0 or $y(R_0)$ blows up. But either term blows up will trigger the blow-up of the other. For example, let $R_0 = \infty$. Hence (3.2) dictates that $\lim_{r\to\infty} G(y(r)) = F(\infty) = \infty$. This is possible only when $y(R_0) = \infty$. Similarly $y(R_0) = \infty$ implies $R_0 = \infty$.

(d) In this case, both $G(\infty)$ and $F(\infty)$ are finite. For any $\alpha > 0$, let y_α denote the rotationally symmetric n-harmonic map such that $y_\alpha(r_0) = \alpha$. (Such n-harmonic solution always exists by solving (1.5). By Theorem 1.1, $y_\alpha(0) = 0$. Since $y'_\alpha(0)$ may not exist, one cannot label the solution by its slope at $r = 0$.) By the uniqueness theorem of solution to initial value problems for ordinary differential equations, solution trajectories corresponding to different values of α cannot intersect one another. Hence for any r_1, $y_{\alpha_2}(r_1) > y_{\alpha_1}(r_1)$ if $\alpha_2 > \alpha_1$, as long as both solutions exist at $r = r_1$.

Define the function h such that $h(\alpha) \equiv \int_\alpha^\infty d\xi/g(\xi)$. As $\alpha \to 0$, $h(\alpha) \to \infty$. On the other hand, when $\alpha \to \infty$, $h(\alpha) \to 0$. Hence there exists a unique $\beta > 0$ such that $h(\beta) = F(\infty)$.

Subcase (d1). Let $\alpha < \beta$. Therefore

$$\int_\alpha^\infty \frac{d\xi}{g(\xi)} > \int_\beta^\infty \frac{d\xi}{g(\xi)} = F(\infty). \tag{3.3}$$

From (3.2),

$$\int_\alpha^{y_\alpha(r)} \frac{d\xi}{g(\xi)} = \int_{r_0}^{r} \frac{d\eta}{f(\eta)}$$

for $r \in (0, R_0)$. We know either $R_0 = \infty$ or $y_\alpha(R_0) = \infty$ in the above equation. As we gradually increase both r and $y_\alpha(r)$, it is clear that r hits ∞ first because of (3.3). Hence $R_0 = \infty$ and y_α is bounded in $\mathbb{R}^+$.

Subcase (d2). Let $\alpha > \beta$. Therefore

$$\int_\alpha^\infty \frac{d\xi}{g(\xi)} < \int_\beta^\infty \frac{d\xi}{g(\xi)} = F(\infty). \tag{3.4}$$

Similar argument as in subcase (d1) shows that R_0 is finite and $y_\alpha(R_0) = \infty$.

Subcase (d3). Let $\alpha = \beta$. Therefore

$$\int_\alpha^\infty \frac{d\xi}{g(\xi)} = \int_\beta^\infty \frac{d\xi}{g(\xi)} = F(\infty). \tag{3.5}$$

Similar argument as in subcase (d1) shows that $R_0 = \infty$ and $y_\alpha(R_0) = \infty$.

The proof of this theorem is now complete.

4. Derivation of integrability using Lie's theory

We now formally derive the integrability of the rotationally symmetric n-harmonic map when $f(r) = r$ using calculus of variation. In order to do so, we need an additional restriction on the class of solutions that we study. This derivation reinforces the common belief that symmetry of a functional will imply the integrability of the resulting Euler–Lagrange equation. Similar treatment on invariant Lagrangian can be found in the literature, for example [7].

Let $L : [0, \infty) \times \mathbf{R}^2 \to \mathbf{R}$, which is defined by

$$L(t, z, p) = t^{n-1}\left(p^2 + (n-1)\frac{g(z)^2}{t^2}\right)^{n/2},$$

and consider the functional

$$I(y)=\int_0^{R_0} L\big(r,y(r),y'(r)\big)\,dr$$

for those functions y such that L is integrable. Since

$$L(r,y,y')=r^{n-1}\left(y'^2+(n-1)\frac{g(y)^2}{r^2}\right)^{n/2}, \tag{4.1}$$

the function $r^{n-1}y'^n$ is integrable. Assume that the functional I is Fréchet differentiable, its critical point will satisfy the Euler–Lagrange equation

$$\frac{d}{dr}\big(L_p(r,y,y')\big)=L_z(r,y,y'). \tag{4.2}$$

It can be verified by direct computation that this is the same as (1.1) when $f(r)=r$.

Next we exploit the symmetry associated with the functional I. Take any $\alpha>0$ and consider the transformation

$$\begin{cases} \tilde r=\alpha r, \\ \tilde y(\tilde r)=y\left(\dfrac{\tilde r}{\alpha}\right)=y(r). \end{cases} \tag{4.3}$$

It can be checked from (4.1) that for all $\alpha>0$,

$$\tilde r L\big(\tilde r,\tilde y(\tilde r),\tilde y'(\tilde r)\big)=\alpha r L\left(\alpha r,y(r),\frac{1}{\alpha}y'(r)\right)=rL\big(r,y(r),y'(r)\big). \tag{4.4}$$

As a consequence, $I(y)=I(\tilde y)$. We expect such a symmetry will have an implication on the integrability of the Euler–Lagrange equation (1.1).

Since $L(r,y(r),y'(r))$ is independent of α, if we differentiate (4.4) with respect to α and then evaluate at $\alpha=1$, we will get

$$rL+r(rL_t-y'L_p)=0, \tag{4.5}$$

where L, L_t and L_p are evaluated at $(r,y(r),y'(r))$. Now define $J=rL-ry'L_p$ where L and L_p are evaluated at $(r,y(r),y'(r))$. Hence for $r>0$,

$$\begin{aligned} \frac{dJ}{dr}&=r(L_t+y'L_z+y''L_p)+L-\left[y'L_p+ry''L_p+ry'\frac{d}{dr}(L_p)\right] \\ &=[L+rL_t-y'L_p]+ry'\left[L_z-\frac{d}{dr}(L_p)\right] \\ &=0 \end{aligned}$$

because of (4.5) and (4.2). Upon integration, we obtain

$$r(L-y'L_p)=\text{constant}\equiv C_1. \tag{4.6}$$

From (4.1), we observe that the term $rL-ry'L_p$ behaves like $(1-n)r^n y'^n$ as $r\to 0+$. Combining this information with (4.6), it is immediate that $r^{n-1}y'^n$ behaves like $\frac{C_1}{(1-n)r}$ as $r\to 0+$. Since $L(r,y,y')\geqslant r^{n-1}y'^n$, the function $L(r,y,y')$ will be integrable only when $C_1=0$. We restrict our attention to this class of solution. Therefore for $r>0$,

$$\begin{aligned} L &= y' L_p \\ &= n r^{n-1} y'^2 \left(y'^2 + (n-1) \frac{g(y)^2}{r^2} \right)^{\frac{n}{2}-1}. \end{aligned}$$

Hence

$$y'^2 + (n-1) \frac{g(y)^2}{r^2} = n y'^2,$$

which implies

$$f y' = g(y)$$

since $f(r) = r$. The formal derivation of integrability using Lie's theory is now complete.

Furthermore we remark that not only Theorem 1.2 provides a rigorous justification of the above intuition, it actually shows that any positive solution y must satisfy $C_1 = 0$ and hence the functional $I(y)$ will be defined.

Acknowledgments

We are grateful to the referee for pointing out the relation between conformal maps and our integrability condition. C.N. Chen and C.K. Law are both partially supported by National Science Council, Taiwan, ROC. The support for C.K. Law is under contract No. NSC 88-2115-M-110-009.

References

[1] K.C. Chang, W.Y. Ding, R.G. Ye, Finite-time blow-up of the heat flow of harmonic maps from surfaces, J. Differential Geom. 36 (1992) 507–515.
[2] C.N. Chen, L.F. Cheung, Y.S. Choi, C.K. Law, On the blowup of heat flow for conformal 3-harmonic maps, Trans. Amer. Math. Soc. 354 (12) (2002) 5087–5110.
[3] L.F. Cheung, C.K. Law, An initial value approach to rotationally symmetric harmonic maps, J. Math. Anal. Appl. 289 (2004) 1–13.
[4] L.F. Cheung, C.K. Law, M.C. Leung, Bounded positive solutions of rotationally symmetric harmonic map equations, Differential Integral Equations 13 (2000) 1149–1188.
[5] L.F. Cheung, C.K. Law, M.C. Leung, An initial value approach to rotationally symmetric p-harmonic maps, 2006, submitted for publication.
[6] L.F. Cheung, C.K. Law, M.C. Leung, J.B. McLeod, Entire solutions of quasilinear differential equations corresponding to p-harmonic maps, Nonlinear Anal. 31 (1998) 701–715.
[7] L. Dreser, Applications of Lie's Theory to Ordinary and Partial Differential Equations, IOP, Bristol, 1999.
[8] R. Greene, H. Wu, Function Theory on Manifolds which Possess a Pole, Lecture Notes in Math., vol. 699, Springer, Berlin, 1970.
[9] J.F. Grotowski, Finite time blow-up for the harmonic map heat flow, Calc. Var. 1 (1993) 231–236.
[10] F. Helein, Harmonic Maps, Conservation Laws and Moving Frames, Cambridge Univ. Press, Cambridge, 2002.
[11] L. D'Onofrio, T. Iwaniec, Notes on p-harmonic analysis, in: Poggi-Corradini (Ed.), The p-Harmonic Equation and Recent Advances in Analysis: 3rd Prairie Analysis Seminar, Amer. Math. Soc., 2003, pp. 25–49.
[12] A. Ratto, M. Rigoli, On the asymptotic behaviour of rotationally symmetric harmonic maps, J. Differential Equations 101 (1993) 15–27.
[13] A. Tachikawa, Rotationally symmetric harmonic maps from a ball into a warped product manifold, Manuscripta Math. 53 (1985) 235–254.
[14] A. Tachikawa, A non-existence result for harmonic mappings from $\mathbb{R}^n$ into $\mathbb{H}^n$, Tokyo J. Math. 11 (1988) 311–316.